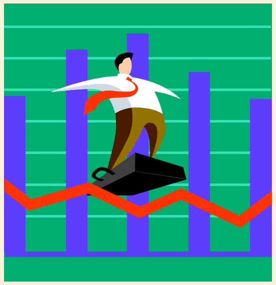
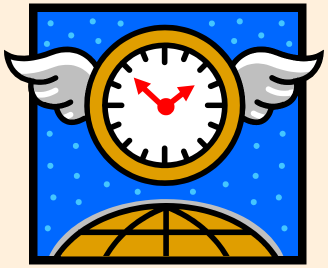
Discounting Axioms Imply Risk Neutrality

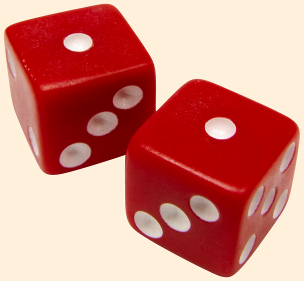
Matthew J. Sobel
Dept. of Operations; Dept. of EE & Computer Sci.
Case Western Reserve University

Major Decisions

- Business, public policy, personal choices
- ***Outcomes uncertain and revealed over time***
- **Stochastic processes**







Background

- Models of intertemporal preference
  - Paul Samuelson
  - **Tjallings C. Koopmans**
  - **A. C. Williams and J. I. Nassar**
- **Intertemporal preference ←→ Risk preference**
- Models of risk preference
  - John von Neumann and Oscar Morgenstern
  - **Israel Herstein and John Milnor**
- **Are time and risk preferences logically independent?**
- **Robert Rosenthal's question**

Outline

- Preferences among stochastic processes
- Important examples
- Axioms
- Preferences among r.v.s
- Discounting theorem
- Risk neutrality theorem
- Composition is NSC for risk neutrality
- Summary

Preferences as Binary Relations

Start with a probability space, and let

V be a real vector space of stochastic processes $X = (X_1, X_2, ...)$

including deterministic sequences of scalars $x = (x_1, x_2, ...)$

with zero element $\theta = (0,0,...)$.

There is a DM (decision maker) whose preferences among stochastic processes are expressed as a binary relation $\succeq$ on V.

Williams-Nassar and Koopmans Examples

EXPECTED PRESENT VALUE:

$X = (X_1, X_2, \ldots, X_n)$ is weakly preferred to $Y = (Y_1, Y_2, \ldots, Y_n)$

$$\text{if} \quad E\left(\sum_{j=1}^{n} \beta_j X_j\right) \geq E\left(\sum_{j=1}^{n} \beta_j Y_j\right)$$

where $\{\beta_j\}$ are *discount* factors.

EXPECTED DISCOUNTED FELICITY:

$X = (X_1, X_2, \ldots, X_n)$ is weakly preferred to $Y = (Y_1, Y_2, \ldots, Y_n)$

$$\text{if} \quad E\left(\sum_{j=1}^{n} \beta_j u(X_j)\right) \geq E\left(\sum_{j=1}^{n} \beta_j u(Y_j)\right)$$

where $u(\cdot)$ is a *felicity* function.

Notation

Sequences of numbers:

$$x = (x_1, x_{2,} ..., x_n) \text{ and } y = (y_1, y_{2,} ..., y_n)$$

The decision maker (DM) weakly prefers x to y: $x \succeq y$ or $y \preceq x$

The DM neither prefers x to y nor y to x (indifference):

$$x \approx y$$

Sequences of random variables:

$$X = (X_1, X_{2,} ..., X_n) \text{ and } Y = (Y_1, Y_{2,} ..., Y_n)$$

The DM weakly prefers X to Y:

$$X \succeq Y$$

The DM neither prefers X to Y nor Y to X (indifference):

$$X \sim Y$$

The DM strongly prefers X to Y ($X \succeq Y$ but not $Y \succeq X$:

$$X \succ Y$$

Axiom Antics

$(A1)$ *Rationality* : $\succeq$ is reflexive, transitive, and complete on V.

$(A2)$ *Decomposition* : $X - Y \succeq \theta \Rightarrow X \succeq Y$

for all $X, Y \in V$.

$(A3)$ *Continuity* : $\{a \in \Re : aX - Y \succeq \theta\}$ is closed for all $X, Y \in V$.

$(A4)$ *Non-triviality* : $(1,0,0,...,0) \succ \theta$.

Decomposition Axiom

$$(A2)\quad \textit{Decomposition}: \quad X - Y \succeq \theta \;\Rightarrow\; X \succeq Y.$$

This is the most controversial and objectionable axiom.

If $\theta \preceq X$ then $(A2)$ implies $\theta \preceq X \preceq 2X \preceq 3X \preceq \cdots$

So $(\preceq, V)$ cannot be consistent with preferences that welcome small gambles but avoid large ones!

Decomposition and Discounting

$$(A2)\quad \textit{Decomposition}: \quad X - Y \succeq \theta \ \Rightarrow\ X \succeq Y$$

$(A2)\ \Leftrightarrow$:

X is as good as the status quo ($X \succeq \theta$) if and only if

for all Y, $\quad X + Y \succeq Y$.

- *(A2) is assumed in **every** axiomatic theory that yields discounting* including Koopmans and Williams-Nassar.

- *(A2)* is not objectionable in deterministic settings.

Decomposition and its Converse

$(A2)$ *Decomposition* : $X - Y \succeq \theta \Rightarrow X \succeq Y$

$(A2) \Leftrightarrow$ [X is as good as the status quo only if incrementing *every* Y with X is at least as good as Y alone:

$X \succeq \theta \Rightarrow X + Y \succeq Y$].

$(A2^C)$ *Composition* : $X \succeq Y \Rightarrow X - Y \succeq \theta$

$(A2^C) \Leftrightarrow$ [X is no better than the status quo if there is *any* Y that is at least as good as Y augmented by X:

if there is any $Y \succeq Y + X$ then $X \preceq \theta$].

Preferences Among R.V.s

Given stochastic processes X and Y and discount factors $\{\beta_j\}$,

the present values $\sum_{j=1}^{n} \beta_j X_j$ and $\sum_{j=1}^{n} \beta_j Y_j$ are random variables.

Let S denote the set of all random variables.

Notice that $(C,0,0,...,0) \in V$ for every $C \in S$.

A preference ordering $\succeq$ on V (among stochastic processes) induces the following preference ordering $\gtrsim$ on S (among random variables):

$$A \gtrsim B \quad \Leftrightarrow \quad (A,0,0,...,0) \succeq (B,0,0,...,0).$$

What properties of $(\gtrsim, S)$ are implied by the axioms for $(\succeq, V)$?

Discounting Theorem

If $(\succeq, V)$ satisfies axioms $(A1)$ - $(A4)$, then there are unique discount factors $\beta_1, \beta_2, ..., \beta_n$ such that

$$X = (X_1, X_2, ..., X_n) \succeq Y = (Y_1, Y_2, ..., Y_n) \quad \Leftrightarrow \quad \sum_{j=1}^{n} \beta_j X_j \gtrapprox \sum_{j=1}^{n} \beta_j Y_j$$

These are the weakest known sufficient conditions for discounting in stochastic *or* deterministic settings.

Discounting Theorem - Proof

If $(\succeq, V)$ satisfies axioms $(A1)$ - $(A4)$, then there are unique discount factors $\beta_1, \beta_2, ..., \beta_n$ such that

$$X = (X_1, X_2, ..., X_n) \succeq Y = (Y_1, Y_2, ..., Y_n) \qquad \Leftrightarrow \qquad \sum_{j=1}^{n} \beta_j X_j \geqq \sum_{j=1}^{n} \beta_j Y_j$$

Most of the proof uses the axioms to build an algebra for $(\succeq, V)$.

Felicity and Utility Functions

A function $u : \mathbb{R} \to \mathbb{R}$ is a *felicity* function for $(\geq\geq, S)$ if $A \geq\geq B \;\Leftrightarrow\; E[u(A)] \geq E[u(B)]$.

A function $w : S \to \mathbb{R}$ is a (von Neumann-Morgenstern) *utility* function if $A \geq\geq B \;\Leftrightarrow w(A) \geq w(B)$.

So if there is a felicity function, then there is a utility function $w(A) = E(u(A)]$.

Axioms Reminder

$(A1)$ *Rationality*: $\succeq$ is reflexive, transitive, and complete on V.

$(A2)$ *Decomposition*: $X - Y \succeq \theta \;\Rightarrow\; X \succeq Y$ for all $X, Y \in V$.

$(A3)$ *Continuity*: $\{a \in \Re : aX - Y \succeq \theta\}$ is closed for all $X, Y \in V$.

$(A4)$ *Non-triviality*: $(1,0,0,\ldots,0) \succ \theta$.

$(A2^C)$ *Composition*: $X \succeq Y \Rightarrow X - Y \succeq \theta$ for all $X, Y \in V$.

Risk Neutrality Theorem

If $(\succeq, V)$ satisfies axioms $(A1)$, $(A2)$, $(A3)$, and $(A4)$, then the following properties are equivalent:

1) $(\succeq, V)$ satisfies the composition axiom, namely $(A2^C)$.

2) There is a felicity function $u(\cdot)$ for $(\geqq, S)$.

3) There are discount factors $\{\beta_j\}$ such that

$$X \succeq Y \quad \Leftrightarrow \quad E\left[\sum_{j=1}^{n} \beta_j X_j\right] \geq E\left[\sum_{j=1}^{n} \beta_j Y_j\right]$$

Risk Neutrality Theorem - proof

If $(\succeq, V)$ satisfies axioms $(A1)$, $(A2)$, $(A3)$, and $(A4)$, then the following properties are equivalent:

1) $(\succeq, V)$ satisfies the composition axiom, namely $(A2^C)$.

2) There is a felicity function $u(\cdot)$ for $(\geqslant, S)$.

3) There are discount factors $\{\beta_j\}$ such that

$$X \succeq Y \quad \Leftrightarrow \quad E\left[\sum_{j=1}^{n} \beta_j X_j\right] \geq E\left[\sum_{j=1}^{n} \beta_j Y_j\right]$$

Key steps in the proof: for all r.v.s A and B,

$$u(A) = -u(-A)$$

$$u(A+B) = u(A) + u(B)$$

Conundrum

If $(\succeq, V)$ satisfies axioms $(A1)$, $(A2)$, $(A3)$, and $(A4)$, then the following properties are equivalent:

1) $(\succeq, V)$ satisfies the composition axiom, namely $(A2^C)$.

2) There is a felicity function $u(\cdot)$ on $(\geqslant, S)$.

3) There are discount factors $\{\beta_j\}$ such that $X \succeq Y \Leftrightarrow E\left[\sum_{j=1}^{n} \beta_j X_j\right] \geq E\left[\sum_{j=1}^{n} \beta_j Y_j\right]$

- **Axioms (A1) – (A4) are *sufficient* for discounting.**
- Are they *necessary* or can they be weakened?
- **With composition, they imply risk neutrality.**
- Can "interesting" preferences satisfy (A1)-(A4) but not $(A2^C)$?

Multiple Attributes

The results remain valid if the components of $X = (X_1, X_2, ...)$ are random vectors instead of real r.v.s.

Recent Result with James Alexander

- If a binary relation on a real vector space satisfies the four axioms, then there is a utility function $f \cdot u$ on V in which $f: R \rightarrow R$ is linear if and only if the binary relation satisfies the composition axiom. That is, composition is necessary and sufficient for risk neutrality (given (A1) – (A4)).

- So if preferences satisfy the four axioms but not composition, then there is a nonlinear felicity function u such that

$$X \succeq Y \quad \Leftrightarrow \quad E[u(\sum \beta_t X_t)] \geq E[u(\sum \beta_t Y_t)]$$

Summary - 1

- Preferences are risk neutral if they satisfy axioms that are the principal justification for discounting with a non-linear felicity function.

  The maximization of $E\left[\sum_j \beta_j u(X_j)\right]$ is self-contradictory

  if $u(\cdot)$ is nonlinear.

- There is a logical basis for discounting without risk neutrality only if the four axioms (A1) – (A4) are satisfied and the composition axiom is *not* satisfied.

Summary - 2

- There is a logical basis for discounting without risk neutrality only if the four axioms (A1) – (A4) are satisfied and the composition axiom is *not* satisfied.
- In that case, there is a slightly stronger logical basis for

$$X \succeq Y \quad \Leftrightarrow \quad E[u(\sum \beta_t X_t)] \geq E[u(\sum \beta_t Y_t)]$$

than for

$$X \succeq Y \quad \Leftrightarrow \quad E[\sum \beta_t u(X_t)] \geq E[u(\sum \beta_t u(Y_t))].$$

Columbia – My Good Fortune

- George Kimball, B. O. Koopman
- Samuel Eilenberg, Serge Lang
- Morton Klein, Pete Veinott
- **Cyrus Derman**